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Periodic Solutions of Duffing Equation

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§ 1 Introduction

The Duffing equation is a simple nonlinear differential equation

$$\frac{d^2x}{dt^2} + \varepsilon\frac{dx}{dt} + bx + cx^3 = P\cos t$$

where ε , b , c are constants and the constant P is the strength of the external force. However it exhibits a big variety of solutions depending on the parameters. If there is no external force, i.e., $P = 0$ and $\varepsilon, c > 0$, $b \geqq 0$, then all the trajectories $(x, dx/dt)$ converge to $(0, 0)$ as $t \to \infty$. It has been proved that if P is not zero, then there exists at least one periodic solution with the same period as the external force. If P and ε are small and $b = 0$, then it is proved that a periodic solution of period 6π exists. It is known that there exist more than one periodic solutions with 2π period for some P and periodic solutions with $4\pi, 6\pi, 8\pi, \cdots,$ period for another P , and even it has non-periodic solutions for other P . These interesting classes of solutions are obtained by analogue computers and by numerical computations of finite difference schemes by digital computers. Ueda[7], Kawakami[4], Thompson-Stewart[6]. There are some analysis on it by Cartwright-Littlewood[1], Yamamoto[10] and so on, but in general a rigourous analysis is not enough powerful to show the existence of these solutions or to clarify the structure, bifurcation, etc. of the solutions because of the nonlinearity.

Here we consider only the periodic solutions and prove that under appropriate conditions periodic solutions with period 2π, 4π, 6π,···, etc. obtained by a finite difference scheme using digital computers are approximate ones to the periodic solutions of Duffing equation, i.e., there exists a real periodic solution with the same period of Duffing equation in a small neighbourhood of the numerical periodic solution. The proof is given by the error estimates for the finite difference schemes and the comutation using the interval operation and by Cesari-Urabe method[3][8]. Our method can be applied also to the van der Pol's equation with the external periodic force. Sinai-Vulkov[5] investigated a similar problem for the nonlinear differential equations without the external force which include Lorenz model by a different method using Poincare mapping and computer.

§ 2 Isolated Periodic Solutions

We remind a nice theory by Cesari[3] and Urabe[8] to guarantee existence of periodic solutions. We look for periodic solutions $x = \hat{x}(t)$ of a system of nonlinear differential equations

$$(2.1) \qquad \frac{dx}{dt} = X(t,x) \quad , \qquad t \in R \quad , \quad x \in D \subset R^n \ ,$$

where X is continuous function with respect to (t,x), is periodic with respect to t with period 2π and is continuously differentiable with respect to x in a bounded open set D.

First we consider a linear system of differential equations

$$\frac{dx}{dt} = A(t)x + f(t) , \quad t \in R , \quad x \in R^n , \tag{2.2}$$

where $A(t)$ is a $n \times n$ matrix whose components are 2π periodic continuous functions with respect to t, and $f(t)$ is a n vector whose components are also 2π periodic continuous functions. Let us denote the fundamental matrix by $\Phi(t)$, i.e.,

$$\frac{d\Phi(t)}{dt} = A(t)\Phi(t) , \quad \Phi(0) = I , \tag{2.3}$$

where I is the identity matrix.

<u>Proposition 1</u> If all the multipliers of linear system (2.3), i.e., all the eigenvalues of the matrix $\Phi(2\pi)$ are different from 1, then the system (2.2) has a unique periodic solution with the same period. It has the representation

$$x(t) = \int_0^{2\pi} H(t,s)f(s)\, ds , \tag{2.4}$$

where H is the Green matrix, i.e.,

$$H = \begin{cases} \Phi(t)(I - \Phi(2\pi)^{-1})\Phi^{-1}(s) , & 0 \leq s < t \leq 2\pi , \\ \Phi(t)(I - \Phi(2\pi)^{-1})\Phi(2\pi)\Phi^{-1}(s) , & 0 \leq t < s \leq 2\pi \end{cases} \tag{2.5}$$

<u>Theorem 2</u> (Urabe) Let us suppose that there exists a 2π periodic continuously differentiable function $x = \bar{x}(t) \subset D$ o: (2.1) which is an approximate solution in the following sense.

(i) It satisfies System (2.1) approximately :

$$(2.6) \qquad \left| \frac{d\bar{x}(t)}{dt} - X(t,\bar{x}(t)) \right| \leqq r \quad \text{for any } t$$

and for a small constant r .

(ii) There exists a $n \times n$ matrix $A = A(t)$ whose components are continuous and 2π periodic with respect to t . All multipliers of (2.3) for A are different from 1 .

(iii) The derivative of the vector X is close to the matrix A , i.e.,

$$(2.7) \qquad | X_x(t,x) - A(t) | \leqq \frac{\kappa}{M} \quad \text{for any } t$$

and for any x such that

$$(2.8) \qquad | x - \bar{x}(t) | \leqq \delta ,$$

where κ $(0,1)$ and δ are some constants, and M is the norm of the Green matrix H for A , i.e.,

$$(2.9) \qquad \| H \|_{2\pi} = 2\pi \max_{k} \sum_{l} | H_{k,l}(t,x) | \leqq M .$$

(iv) The constants r, κ, M and δ satisfy the condition

$$(2.10) \qquad \frac{M\, r}{1-\kappa} \leqq \delta .$$

Then there exists a unique 2π periodic solution $x = \hat{x}(t)$ of the nonlinear system (2.1) in the neighbourhood of $x = \bar{x}(t)$.

$$(2.11) \qquad | \hat{x}(t) - \bar{x}(t) | \leqq \frac{M\, r}{1-\kappa} \quad \text{for any } t .$$

This is a kind of Newton method to obtain the periodic solution and it is proved by Urabe[8] following Cesari[3]. It is worth to notice that it is powerful for small solutions. Because in the case $\overline{x}(t) = 0$, $A(t)$ = constant and the analysis and estimates for (i)-(iv) are rather easy. As another application Urabe-Reiter[9] used his theorem to suggest numerically the existence of periodic solutions for several systems of differential equations using Galerkin method and computers. However their error estimates and bounds are rather crude to guarantee the existence of periodic solutions. In the following we use his theorem to prove existence of periodic solutions of Duffing equation by a finite difference scheme, interval operations using computer and precise error estimates.

§ 3 Finite Differnce Scheme

Duffing equation can be written in a system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -\varepsilon y - (b + cx^2)x + P(t) \end{pmatrix} . \tag{3.1}$$

Let us define a finite difference scheme to solve Duffing equation with 2π periodic force term $P(t)$.

$$\Delta t = \frac{2\pi}{N} , \tag{3.2}$$

where we will choose $10^3 \leqq N \leqq 10^5$ later. When the values x_k, y_k, P_k at $t = k\Delta t$ for some k are known, we define the values z_k, w_k, v_k and u_k by

$$(3.3)\quad \begin{cases} z_k = -\varepsilon y_k - bx_k - cx_k^3 + P_k , \\ w_k = -\varepsilon z_k - (b + 3cx_k^2)y_k + \dot{P}_k , \\ v_k = -\varepsilon w_k - (b + 3cx_k^2)z_k - 6cx_k y_k^2 + \ddot{P}_k , \end{cases}$$

$$u_k = -\varepsilon v_k - (b + 3cx_k^2)w_k - 18cx_k y_k z_k - 6cy_k^3 + \dddot{P}_k ,$$

where

(3.4) $\quad P_k = P(k\Delta t) , \quad \dot{P}_k = dP(k\Delta t)/dt$ and so on.

Our finite difference scheme is a simple explicit one

$$(3.5) \quad \begin{cases} x_{k+1} = x_k + \Delta t y_k + \dfrac{\Delta t^2}{2} z_k + \dfrac{\Delta t^3}{6} w_k + \dfrac{\Delta t^4}{24} v_k , \\ y_{k+1} = y_k + \Delta t z_k + \dfrac{\Delta t^2}{2} w_k + \dfrac{\Delta t^3}{6} v_k + \dfrac{\Delta t^4}{24} u_k . \end{cases}$$

Then the formal accuracy of the scheme is $O(\Delta t^4)$.

Now we carry out numerical computations using the scheme (3.5) in double precision from an initial data $(\bar{x}_0, \bar{y}_0)$ for a period $k = 1,2,\cdots,N$, while we store the values for $k = 0,1,2,\cdots,N$ of each period . We examine the period error :

$$\max \{ |\bar{x}_0 - \bar{x}_N|, |\bar{y}_0 - \bar{y}_N| \}$$

for the periodicity. If the period error is not small, we change $\bar{x}_0 = \bar{x}_N$, $\bar{y}_0 = \bar{y}_N$ and make computations for $k = 1,2,\cdots,N$ again and so on. We run the computer until the period error becomes less than 0.9×10^{-13} . When the period error becomes less than 0.9×10^{-13} :

(3.6) $\quad$ period error = $\max \{ |\bar{x}_0 - \bar{x}_N|, |\bar{y}_0 - \bar{y}_N| \} < 0.9\times10^{-13}$,

we stop the computation and use the computed values

(3.7) $\quad (\bar{x}_0, \bar{y}_0), (\bar{x}_1, \bar{y}_1), \cdots\cdots, (\bar{x}_N, \bar{y}_N)$

to make continuous periodic functions as follows.

$$(3.8)\begin{cases} \bar{x}(t) = \bar{x}_k + (t-k\Delta t)\bar{y}_k + \dfrac{(t-k\Delta t)^2}{2}\bar{z}_k + \dfrac{(t-k\Delta t)^3}{6}\bar{w}_k + \\ \qquad + \dfrac{(t-k\Delta t)^4}{24}\bar{v}_k + \dfrac{(t-k\Delta t)^5}{\Delta t^5}\alpha_{k+1} \quad , \\ \bar{y}(t) = \bar{y}_k + (t-k\Delta t)\bar{z}_k + \dfrac{(t-k\Delta t)^2}{2}\bar{w}_k + \dfrac{(t-k\Delta t)^3}{6}\bar{v}_k + \\ \qquad + \dfrac{(t-k\Delta t)^4}{24}\bar{u}_k + \dfrac{(t-k\Delta t)^5}{\Delta t^5}\beta_{k+1} \quad , \end{cases}$$

$$\text{in} \quad k\Delta t \leqq t < (k+1)\Delta t \ , \quad \text{for} \quad k = 0,1,\cdots,N-1 \ .$$

Here $\bar{z}_k$, $\bar{w}_k$, $\bar{v}_k$, $\bar{u}_k$ are computed values by (3.3) with $x_k = \bar{x}_k$, $y_k = \bar{y}_k$. Also the last terms α_{k+1} and β_{k+1} are defined by the relations

$$(3.9)\begin{cases} \bar{x}_{k+1} = \bar{x}((k+1)\Delta t) \ , \quad \bar{y}_{k+1} = \bar{y}((k+1)\Delta t) & \text{for} \quad k = 0,1,\cdots,N-2 \\ \bar{x}_0 = \bar{x}(N\Delta t) \ , \quad \bar{y}_0 = \bar{y}(N\Delta t) \ , & \text{for} \quad k = N-1 \end{cases}$$

and so they are numerical errors because of digital computer using double precision for (3.2)(3.3)(3.4)(3.5). By this adjustment the functions $\bar{x}(t)$, $\bar{y}(t)$ are 2π periodic continuous and piecewise continuously differentiable functions attaining the computed values $\bar{x}_k$, $\bar{y}_k$ at time $t = k\Delta t$ for $k = 0,1,\cdots,N$.

We have to estimate the numerical errors α_k, β_k, which may be expected less than the periodic error 0.9×10^{-13} because we have used computations in double precision which has approximately the precision of 10^{-16}. In fact we can check this numerical error for each k by using machine interval operations for the computations (3.2), (3.3) with $x_k = \bar{x}_k$, $y_k = \bar{y}_k$, (3.4) and (3.5).

Namely let us use the same computer with the software which can carry out machine interval operations. Let (x) denote the interval containing a real number x which is the minimum interval expressed by the double precision floating point number of the computer. Then a machine interval operation of machine intervals gives also a machine interval, which contains the exact value by the real number operations in it and also does the value obtained by the usual machine computation of double precision floating point operations in it. Let us denote

$$(3.10)\quad \begin{aligned} &(\Delta t) = (2\pi)/N \quad , \\ &(z_k) = -(\varepsilon)\overline{y}_k - (b)\overline{x}_k -(c)\overline{x}_k^{\,3} + (P_k) \quad , \text{ etc.,} \end{aligned}$$

where each operation is interpreted as an machine interval operation. Thus the computed value by double precision is in the interval

$$\overline{\Delta t} \in (\Delta t)$$

and the numerical error is the length of the interval (Δt) . We will see it for each computation example treated later that we have the interval for the value x_{k+1}, y_{k+1} following (3.5) whose length is less than 10^{-14} . Therefore we know

$$(3.11)\qquad |\alpha_k| \ , \ |\beta_k| \ < \ 10^{-14} \quad , \quad k = 1,2,\cdots,N-1 \ .$$

Here we notice that by the definition $(\overline{x}_k, \overline{y}_k)$ for $k = 1,\cdots,N-1$ in (3.7) and $(\overline{x}(t), \overline{y}(t))$ in (3.8) the interval analysis is necessary for only each step k . At the last step from $k = N-1$ to $k = N$ by the numerical error 10^{-14} and the period error 0.9×10^{-13} we have

$$(3.12) \qquad |\alpha_N| \,,\ |\beta_N| \ < \ 10^{-13} \,.$$

These error estimates (3.11)(3.12) are the most essential part where the interval analysis is used in the method of our paper. In this way we have obtained a continuous and piecewise continuously differentiable periodic functions $(\bar{x}(t),\ \bar{y}(t))$, which will be regarded as our approximation for periodic solution. Then we have to notice that Theorem 2 is still valid even if the periodic approximate solution $\bar{x}(t)$ is continuous and piecewise continuously differentiable function. We proceed to estimate the equation error r as an approximate solution of $(\bar{x}(t),\ \bar{y}(t))$. It is easy to see that on each interval $k\Delta t \leqq t < (k+1)\Delta t$

$$\left|\frac{d\bar{x}(t)}{dt} - \bar{y}(t)\right| = \left|\frac{(t-k\Delta t)^4}{-24}u_k + \frac{5(t-k\Delta t)^4}{\Delta t^5}\alpha_{k+1} + \frac{(t-k\Delta t)^5}{\Delta t^5}\beta_{k+1}\right|$$

$$\leqq \Delta t^3\left\{\frac{\Delta t}{24}|u_k| + \frac{5|\alpha_{k+1}|}{\Delta t^4} + \frac{|\beta_{k+1}|}{\Delta t^3}\right\} = E_0\Delta t^3 \,.$$

(3.13)

$$\left|\frac{d\bar{y}(t)}{dt} + \varepsilon\bar{y}(t) + b\bar{x}(t) + c\bar{x}(t)^3 - P(t)\right|$$

$$\leqq \Delta t^3\left\{\frac{\Delta t}{24}\left(R_k + \frac{\Delta t}{5}S_k\right) + \left|\frac{d^4P(\tau)}{dt^4}\right| + T_k\right\} = E_1\Delta t^3 \,,$$

where

(3.14)

$$R_k = \left|\varepsilon\bar{u}_k + b\bar{v}_k + c(18\bar{x}_k\bar{z}_k^{\,2} + 24\bar{x}_k\bar{y}_k\bar{w}_k + 3\bar{x}_k^{\,2}\bar{v}_k + 36\bar{y}_k\bar{z}_k)\right| \,,$$

$$S_k = \left\{\left|20\bar{z}_k\bar{w}_k + 10\bar{y}_k\bar{v}_k\right| + \frac{\Delta t}{6}\left|20\bar{w}_k^{\,2} + 30\bar{z}_k\bar{v}_k\right|\right.$$

$$\left. + \frac{\Delta t^2}{6\cdot 7}\left|70\bar{w}_k\bar{v}_k\right| + \frac{\Delta t^3}{6\cdot 7\cdot 8}\left|70\bar{v}_k^{\,2}\right|\right\}|\bar{x}(t)| +$$

(3.15)

$$+ \left| 20\bar{x}_k\bar{y}_k\bar{v}_k + 60\bar{y}_k^{\,2}\bar{w}_k + 40\bar{x}_k\bar{z}_k\bar{w}_k + 90\bar{y}_k\bar{z}_k^{\,2} \right| ,$$

$$T_k = \frac{5\beta_{k+1}}{\Delta t^4} + \frac{\varepsilon\beta_{k+1}}{\Delta t^3} + \frac{b\alpha_{k+1}}{\Delta t^3} +$$

(3.16)

$$+ \frac{c\alpha_{k+1}}{\Delta t^3}(3\bar{x}(t)^2 + 3|\bar{x}(t)|\alpha_{k+1} + \alpha_{k+1}{}^2) .$$

Before obtaining the bound for E_0 and E_1 we get the estimate for the maximum of the absolute value of $\bar{x}(t)$ and $\bar{y}(t)$ on each interval $k\Delta t \leqq t < (k+1)\Delta t$, which is obtained in the following steps.

(i) We take the absolute value of $\bar{x}_k$, $\bar{y}_k$, (z_k), (w_k), (v_k), (u_k), where (z_k) denotes the interval computed by (3.3) for $x = \bar{x}_k$, $y = \bar{y}_k$, $P_k = (P_k)$ and so on. Thus we have

(3.17) $|\bar{x}_k|$, $|\bar{y}_k|$, $|(z_k)|$, $|(w_k)|$, $|(v_k)|$, and $|(u_k)|$,

where the notation $|(z_k)|$ means the maximum of the absolute values of both ends of the interval (z_k).

(ii) Estimate for $\bar{x}(t)$ and $\bar{y}(t)$. If we compute

$$(X) = |\bar{x}_k| + (\Delta t)|\bar{y}_k| + \frac{(\Delta t)^2}{2}|(z_k)| + \frac{(\Delta t)^3}{6}|(w_k)| +$$

$$+ \frac{(\Delta t)^4}{24}|(v_k)| + |\alpha_{k+1}| ,$$

(3.18)

$$(Y) = |\bar{y}_k| + (\Delta t)|z_k| + \frac{(\Delta t)^2}{2}|(w_k)| + \frac{(\Delta t)^3}{6}|(v_k)| +$$

$$+ \frac{(\Delta t)^4}{24}|(u_k)| + |\beta_{k+1}| ,$$

then we have on each interval $k\Delta t \leqq t < (k+1)\Delta t$

(3.19) $\quad |\bar{x}(t)| \leqq |(X)| \quad , \quad |\bar{y}(t)| \leqq |(Y)| \quad .$

Now we compute by interval operations similarly for (3.14)(3.15) (3.16), where $\bar{x}(t)$ is replaced by (X) and $\bar{z}_k$ is by (z_k) and so on, to obtain

(3.20) $\quad (R_k),\ (S_k),\ (T_k)$

which bound $|R_k|,\ |S_k|,\ |T_k|$.

Then we have the estimate for the equation error r for (2.6) from (3.11)(3.12)(3.13)(3.20).

(3.21) $\quad r = \max \{ E_0\Delta t^3 \ , \ E_1\Delta t^3 \} \quad ,$

where E_0, E_1 are defined in (3.13).

§ 4 Linearized Equation

We consider the linearized differential equation about the 2π periodic approximate solution $\bar{x}(t)$.

(4.1) $$\frac{d\phi}{dt} = A(t)\phi \quad , \quad \phi(0) = \phi_0 \quad ,$$

where $A(t)$ is the linearization of Duffing equation (3.1) about $x = \bar{x}(t)$, i.e.,

(4.2) $$A(t) = \begin{pmatrix} 0 & 1 \\ -b-3ca(t)^2 & -\varepsilon \end{pmatrix} \quad ,$$

where $a(t) = \bar{x}(t)$ is a 2π periodic continuous and piecewise continuously differentiable function which is obtained by (3.8).

The solution $\phi(t)$ of (4.1) is also continuous and piecewise continuously differentiable, and the fourth derivative satisfies the equation on each time interval $k\Delta t \leqq t < (k+1)\Delta t$, $k = 0,1,\cdots,N-1$,

$$\frac{d^4\phi}{dt^4} = D(t)\phi \quad , \tag{4.3}$$

where

$$D(t) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \quad , \tag{4.4}$$

and

$$\begin{cases} D_{11} = -\varepsilon^2(b+3ca^2(t)) + (b+3ca^2(t))^2 + 6\varepsilon ca(t)\dot{a}(t) - \\ \qquad - 6c\dot{a}(t)^2 - 6ca(t)\ddot{a}(t) \quad , \\ D_{12} = -\varepsilon^3 + 2\varepsilon(b+3ca^2(t)) - 12ca(t)\dot{a}(t) \quad , \\ D_{21} = \varepsilon^3(b+3ca^2(t)) - 2\varepsilon(b+3ca^2(t))^2 + 24bca(t)\dot{a}(t) + \\ \qquad + 72c^2a^3(t)\dot{a}(t) - 6\varepsilon^2ca(t)\dot{a}(t) + 6\varepsilon c\dot{a}(t)^2 + \\ \qquad + 6\varepsilon ca(t)\ddot{a}(t) - 18\varepsilon c\dot{a}(t)\ddot{a}(t) - 6ca(t)\dddot{a}(t) \quad , \\ D_{22} = \varepsilon^4 - 3\varepsilon^2(b+3ca^2(t)) + 30\varepsilon ca(t)\dot{a}(t) - \\ \qquad - 18c\dot{a}(t)^2 - 18ca(t)\ddot{a}(t) + (b+3ca^2(t))^2 \quad , \end{cases} \tag{4.5}$$

where $\dot{a}(t) = da(t)/dt$, etc.

As the function $a(t) = \overline{x}(t)$ given by (3.8) is a polynomial on each time interval $k\Delta t \leqq t < (k+1)\Delta t$, all D_{ij} are polynomials of t on the interval. However they have rather too many terms to obtain the explicit form and the best bound for the norm of matrix D . In order to estimate the matrix norm of D we write the function $a(t)$ in (3.8) and its derivatives in the following forms.

$$(4.6)\quad \begin{cases} a(t) = x + ty + t^2A_0 , & \dot{a}(t) = y + tz + t^2A_1 , \\ \ddot{a}(t) = z + tw + t^2A_2 , & \dddot{a}(t) = w + tv + t^2A_3 , \end{cases}$$

where x, y, z, w, v stand for $\bar{x}_k, \bar{y}_k, \bar{z}_k, \bar{w}_k, \bar{v}_k$ and t does for $t-k\Delta t$, and A_i, $i= 0,1,2,3,$ contains the remaining polynomials. Thus they have the estimates.

$$(4.7)\quad \begin{cases} |A_0| \leqq \max\limits_{0\leqq t\leqq \Delta t} \left| \frac{1}{2}\bar{z}_k + \frac{t}{6}\bar{w}_k + \frac{t^2}{24}\bar{v}_k + \frac{t^3}{\Delta t^5}\alpha_{k+1} \right| \\ \qquad \leqq \frac{1}{2}|\bar{z}_k| + \frac{\Delta t}{6}|\bar{w}_k| + \frac{\Delta t^2}{24}|\bar{v}_k| + \frac{1}{\Delta t^2}|\alpha_{k+1}| , \\ |A_1| \leqq \frac{1}{2}|\bar{w}_k| + \frac{\Delta t}{6}|\bar{v}_k| + \frac{5}{\Delta t^3}|\alpha_{k+1}| , \\ |A_2| \leqq \frac{1}{2}|\bar{v}_k| + \frac{20}{\Delta t^4}|\alpha_{k+1}| , \\ |A_3| \leqq \frac{60}{\Delta t^5}|\alpha_{k+1}| . \end{cases}$$

Substituting (4.6) in (4.5) we have the expressions

$$(4.8)\quad D_{ij} = X_{ij} + tY_{ij} + t^2Z_{ij} + t^2R_{ij} , \quad i, j = 1, 2 ,$$

where R_{ij} contains the terms A_0, A_1, A_2 and A_3 . Then we obtain the bounds for D_{ij} as follows.

$$(4.9)\quad |D_{ij}| \leqq |X_{ij}| + \Delta t|Y_{ij}| + \Delta t^2|Z_{ij}| + \Delta t^2|R_{ij}| ,$$

where we use the bounds (4.7) of A_i for $|R_{ij}|$.

From this evaluation of $|D_{ij}|$ we can compute a bound for the matrix norm

(4.10) $$\|D\| = \max\{\max_k |D_{11}|+|D_{12}|, \max_k |D_{21}|+|D_{22}|\} .$$

Now we turn to a computation of the fundamental matrix $L(t,s)$, $0 \leqq s \leqq t \leqq 2\pi$, for the linear equation (4.1)(4.2). Let us define the finite difference scheme for

(4.11) $$\Delta t = \frac{2\pi}{N}, \quad 0 \leqq m \leqq k \leqq n \leqq N ,$$

as follows. When Φ_k and a_k, $\dot{a}_k$, $\ddot{a}_k$ are known, we define the matrix

(4.12) $$\begin{cases} A_k = A(k\Delta t) = \begin{pmatrix} 0 & 1 \\ -b-3ca_k^2 & -\varepsilon \end{pmatrix} , \\ B_k = \dot{A}(k\Delta t) + A(k\Delta t)^2 , \\ C_k = \ddot{A}_k(k\Delta t) + 2\dot{A}(k\Delta t)A(k\Delta t) + A(k\Delta t)\dot{A}(k\Delta t) + A(k\Delta t)^3 , \end{cases}$$

where we remember $a(t) = \bar{x}(t)$, $a_k = a(k\Delta t)$, $\dot{a}_k = \dot{a}(k\Delta t)$, $\ddot{a}_k = \ddot{a}(k\Delta t)$.

Our finite difference scheme is a simple explicit one :

(4.13) $$\begin{cases} \Phi_{k+1} = \Phi_k + \Delta t A_k \Phi_k + \dfrac{\Delta t^2}{2} B_k \Phi_k + \dfrac{\Delta t^3}{6} C_k \Phi_k , \\ \qquad\qquad k = m, m+1, \cdots, n-1 , \\ \Phi_m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \end{cases}$$

The formal accuracy of the scheme is $O(\Delta t^3)$ and the error is given by (4.3) and (4.4).

Now we carry out numerical computations using the scheme (4.13) in double precision. Let us denote the computed values by $\bar{\Phi}_k$, $k = m, m+1, \cdots, n$, and compare them with the finite difference solution Φ_k and also with the fundamental matrix $L(t,s)$, $s = m\Delta t$, $t = k\Delta t$, following the method of Losinsky (cf. [5]).

Proposition 4.1

$$|\, L(n\Delta t, m\Delta t) - \bar{\Phi}_n \,| \leq 2\pi C_1 (1 + C_1 C_3 \frac{\Delta t}{24})\, \Delta t^2 \quad , \tag{4.14}$$

where

$$C_1 = \max_{s \leq t} |\, L(t,s) \,| \quad , \tag{4.15}$$

and the constant C_3 is the bound for

$$\left| \frac{d^4 \Phi(t)}{dt^4} \right| \leq C_3 \,|\, \Phi(t) \,| \quad , \tag{4.16}$$

given by (4.10).

This proposition is crucial to estimate the norm M in (2.9). The detailed proof and examples by the above method using computer which prove the existence of periodic solutions of the duffing equation will be given elsewhere.

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